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# The evolution operator of the Hartree-type equation with a quadratic potential

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**Abstract**
Based on the ideology of the Maslov complex germ theory, a method has been developed for finding an exact solution of the Cauchy problem for a Hartree-type equation with a quadratic potential in the class of semiclassically concentrated functions. The nonlinear evolution operator has been obtained in explicit form in the class of semiclassically concentrated functions. Parametric families of symmetry operators have been found for the Hartree-type equation. With the help of symmetry operators, families of exact solutions of the equation have been constructed. Exact expressions are obtained for the quasi-energies and their respective states. The Aharonov–Anandan geometric phases are found in explicit form for the quasi-energy states.

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## Introduction

The integration of nonlinear equations of mathematical physics is known to be a fundamental problem. For some classes of nonlinear equations, families of particular solutions can be constructed using symmetry analysis methods (see [1–6] and the references therein). Of special interest are equations in multidimensional spaces with variable coefficients. The methods of exact integration of equations of this class are few in number. Moreover, only a few particular cases of solving such equations are known, which, nevertheless, are of considerable interest in view of the complexity of the problem. For example, for a Hartree-type equation of special form, the methods of construction of particular solutions are discussed in [7]. It appears that approximate (asymptotic) methods work well in studying the classes of these types of equations.

A method of construction of semiclassical asymptotics for a Hartree-type equation with smooth coefficients and a cubic nonlocal nonlinearity has been developed [8, 9] based on the Maslov complex germ theory [10, 11]. The key point of the method is the integration of an

auxiliary system of ordinary differential equations—a Hamilton–Ehrenfest system [12]. The solutions of this system allow one to construct the associated linear Schrödinger equation. The solutions of this equation, in turn, make it possible to find, with a given accuracy, solutions of the Hartree-type equation.

Besides the well-known applications in quantum mechanics (see, e.g., [13, 14]), the Hartree-type equation is a base equation in constructing models which describe the Bose–Einstein condensate (see, e.g., the review [15]), where this equation is called the Gross–Pitaevskii equation.

In the present work, the Cauchy problem for the one-dimensional Hartree-type equation with a quadratic potential is solved in the class of the semiclassically concentrated functions:

$$\{-i\hbar\partial_t + \widehat{\mathcal{H}}(t) + \varkappa\widehat{V}(t, \Psi)\}\Psi = 0$$

$$\widehat{\mathcal{H}}(t) = \frac{\hat{p}^2}{2m} + \frac{kx^2}{2} - eEx\cos\omega t \tag{0.1}$$

$$\widehat{V}(t, \Psi)\Psi = \frac{1}{2}\int_{-\infty}^{+\infty} dy[ax^2 + 2bxy + cy^2]|\Psi(y, t)|^2\Psi(x, t).$$

Here, $k > 0, m, e, E, a, b$ and $c$ are the parameters of the potential, $\varkappa$ is the nonlinearity parameter. This equation is of independent value for applications since, for example, an external field specified by a quadratic potential is used to describe magnetic traps in models of the Bose–Einstein condensate. With this simple example, it is possible to illustrate in detail the basic ideas of the method proposed in [8, 9]. It should be stressed that for the case under consideration this method gives an *exact* solution of the Cauchy problem for the Hartree-type equation in the class of semiclassically concentrated functions. As a result, for a nonlinear Hartree-type equation it is possible to find in explicit form not only the evolution operator, but also the symmetry operators in the functional space under consideration.

The proposed approach constructively extends the area of application of the group analysis to the case of quantum-mechanical integro-differential equations of the form (0.1). Symmetry operators were considered in [16] for a special class of differential equations in the framework of the $L$-transformation method (via a transformation to the Lagrange variables). For these nonlinear equations, one-parametric families of symmetry operators have also been constructed using $L$-transformations and the invariance group of the equation [16].

It should be noted that all basic statements and constructions of the present work remain valid, with the prescribed accuracy in $\hbar$, $\hbar \to 0$, for the Hartree-type equation of general form. A feature of the case under consideration is that all basic statements can be checked by direct substitution, which we have just done.

## 1. The Hamilton–Ehrenfest system of equations

The class of trajectory concentrated functions is a key point in the asymptotic integration method proposed in [8, 9] for the Hartree-type equation. Since the latter is a quantum mechanical equation, the problem of correct mathematical extraction of classical equations of motion from quantum ones, posed by Ehrenfest [17], is to be studied in view of the general principles of quantum mechanics. In respect to the correspondence principle, the classical equations themselves enter the complete quantum description of a system. One of the common approaches used to attack this problem in the standard linear quantum mechanics (see, e.g., [14]) is to deduce a Hamilton–Jacobi equation from the Schrödinger equation taking the formal limit $\hbar \to 0$ ($\hbar$ is the Planck constant). To obtain solutions of the classical Hamilton equations from the Hamilton–Jacobi equation, one has to introduce the notion of the phase trajectory of a classical system. A mathematical framework for this approach was developed in [18].

For the Hartree-type equation, it proves to be problematic to obtain, in the above sense, the Hamilton–Jacobi equation for a classical action. Another way to obtain 'classical' equations is to directly derive them from the Hartree-type equation. To this end, a phase trajectory is to be introduced in quantum mechanics. Let the state vector of a system be $\Psi$ and $\hat{x}$ and $\hat{p}$ describe a set of generalized coordinate operators and conjugated momenta. Then

$$[\hat{x}, \hat{p}] = i\hbar. \tag{1.1}$$

The quantum averages with respect to $\Psi$

$$\langle \hat{x} \rangle = \langle \Psi | \hat{x} | \Psi \rangle \qquad \langle \hat{p} \rangle = \langle \Psi | \hat{p} | \Psi \rangle \tag{1.2}$$

are functions of time and depend parametrically on $\hbar$

$$\langle \hat{x} \rangle = x_\Psi(t, \hbar) \qquad \langle \hat{p} \rangle = p_\Psi(t, \hbar). \tag{1.3}$$

Here, $\langle \Psi | \Phi \rangle$ is a scalar product in $L_2(\mathbb{R})$. If there exists the limit

$$\lim_{\hbar \to 0} x_\Psi(t, \hbar) = X(t) \qquad \lim_{\hbar \to 0} p_\Psi(t, \hbar) = P(t) \tag{1.4}$$

then the quantities $x = X(t)$, $p = P(t)$ are naturally to be called the phase trajectory of the classical system corresponding to the state $\Psi$. Clearly, both average values (1.3) and the limit values (1.4) depend on the state $\Psi$. Consequently, the condition that (1.4) is a solution of the classical equations of motion is a constraint on the state $\Psi$. It is natural to speculate the solutions of the Hartree-type equation that admit this limit to be close to classical solutions (below we are concerned only with these solutions). Otherwise the solutions to the Hartree-type equation are to be considered as essentially quantum. Obviously, $|\Psi(x, t, \hbar)|^2$ is to tend to $\delta(x - X(t))$ as $\hbar \to 0$ for the states of the first class. A similar conclusion can be made for the wavefunction $\Psi$ in the $p$-representation $\widetilde{\Psi}(p, t, \hbar)$, that is,

$$\lim_{\hbar \to 0} |\Psi(x, t, \hbar)|^2 = \delta(x - X(t)) \tag{1.5}$$

$$\lim_{\hbar \to 0} |\widetilde{\Psi}(p, t, \hbar)|^2 = \delta(p - P(t)). \tag{1.6}$$

Let us require for a solution of equation (0.1) in addition to the fulfilment of conditions (1.5) and (1.6), the existence of moments of any order. Then it is natural to seek a solution of equation to (0.1) in the form of the following ansatz:

$$\Psi(x, t, \hbar) = \varphi\left(\frac{\Delta x}{\sqrt{\hbar}}, t, \sqrt{\hbar}\right) \exp\left[\frac{i}{\hbar}(S(t, \hbar) + P(t)\Delta x)\right]. \tag{1.7}$$

Here, the function $\varphi(\xi, t, \sqrt{\hbar}) \in \mathbb{S}$ ($\mathbb{S}$ is Schwartz's space) with respect to the variable $\xi = \Delta x / \sqrt{\hbar}$ and regularly depends on $\sqrt{\hbar}$, and $\Delta x = x - X(t)$. The real functions $S(t, \hbar)$, $Z(t) = (P(t), X(t))$, which characterize the solution, are to be determined. We shall designate the class of functions (1.7) by the symbol $\mathcal{P}_\hbar^t$ and call it *the class of semiclassically concentrated functions* (see [8]).

Consider the Cauchy problem

$$\Psi(x, t, \hbar)|_{t=s} = \psi(x, \hbar) \qquad \psi(x, \hbar) \in \mathcal{P}_\hbar^0 \tag{1.8}$$

for equation (0.1).

For a linear operator $\widehat{A}$, the average value $\langle \widehat{A} \rangle$ in the state $\Psi(x, t, \hbar)$ is defined as

$$\langle \widehat{A} \rangle = \frac{1}{\|\Psi(t)\|^2} \langle \Psi(t) | \widehat{A} | \Psi(t) \rangle = A_\Psi(t, \hbar). \tag{1.9}$$

For average values of the operator $\widehat{A}$ on the solutions $\Psi(t)$ of equation (0.1) we have

$$\frac{d\langle \widehat{A}(t) \rangle}{dt} = \left\langle \frac{\partial \widehat{A}(t)}{\partial t} \right\rangle + \frac{i}{\hbar} \langle [\widehat{\mathcal{H}}_\varkappa(t, \Psi(t)) \widehat{A}(t)] \rangle \tag{1.10}$$

where $[\widehat{A}, \widehat{B}] = \widehat{A}\widehat{B} - \widehat{B}\widehat{A}$ is the commutator of the linear operators $\widehat{A}, \widehat{B}$.

By analogy with the quantum mechanical linear Schrödinger equation, we shall call relation (1.10) the Ehrenfest equation [17]. From the Ehrenfest equation, in particular, for $\widehat{A} = 1$, it follows that the norm of the solution of equation (0.1) does not depend on time, that is, we have $\|\Psi(t)\|^2 = \|\Psi(0)\|^2 = \|\Psi\|^2$. As a result, we can pass in equation (0.1), without loss of generality, from the constant $\varkappa$ to the constant $\tilde{\varkappa} = \varkappa \|\Psi\|^2$.

Let us designate by

$$\alpha_\Psi^{(l,j)}(t,\hbar) = \frac{1}{\|\Psi\|^2} \int_{-\infty}^{+\infty} \Psi^*(x,t)\{(\Delta\hat{p}_x)^l(\Delta x)^j\}\Psi(x,t)\,dx \qquad j,l = \overline{0,\infty} \qquad (1.11)$$

the $(j+l)$-order moments of the function $\Psi(x,t)$ centred about $x_\Psi(t,\hbar)$, $p_\Psi(t,\hbar)$. Here, $\Delta\hat{p}_x = -i\hbar\partial_x - p_\Psi(t,\hbar)$, $\Delta x = x - x_\Psi(t,\hbar)$ and $\{(\Delta\hat{p}_x)^l(\Delta x)^j\}$ is a Weyl-ordered operator with the symbol $(\Delta p_x)^l(\Delta x)^j$. Along with designations (1.11), we shall use for the variances of coordinates and moments and for the correlation functions of coordinates and moments the following designations:

$$\sigma_{xx}(t,\hbar) = \alpha_\Psi^{(0,2)}(t,\hbar) \qquad \sigma_{pp}(t,\hbar) = \alpha_\Psi^{(2,0)}(t,\hbar) \qquad \sigma_{xp}(t,\hbar) = \alpha_\Psi^{(1,1)}(t,\hbar).$$

Let us write the Ehrenfest equations for the average values of the operators $\hat{p}$, $x$, $\{(\Delta x)^k(\Delta\hat{p})^l\}$, $k,l = \overline{1,M}$. As a result, we obtain for the first-order and second-order moments the following system of equations:

$$\begin{cases} \dot{p} = -m\big(\omega_0^2 + \zeta(a+b)\omega_{\mathrm{nl}}^2(a+b)\big)x + eE\cos\omega t \\ \dot{x} = \dfrac{p}{m} \end{cases} \qquad (1.12)$$

$$\begin{cases} \dot{\sigma}_{xx} = \dfrac{2}{m}\sigma_{xp} \\ \dot{\sigma}_{xp} = \dfrac{1}{m}\sigma_{pp} - m\big(\omega_0^2 + \zeta(a)\omega_{\mathrm{nl}}^2(a)\big)\sigma_{xx} \\ \dot{\sigma}_{pp} = -2m\big(\omega_0^2 + \zeta(a)\omega_{\mathrm{nl}}^2(a)\big)\sigma_{xp}. \end{cases} \qquad (1.13)$$

Similarly, for the higher-order moments, we find

$$\begin{cases} \dot{\alpha}^{(j,l)} = \dfrac{l}{m}\alpha^{(j+1,l-1)} - jm\big(\omega_0^2 + \zeta(a)\omega_{\mathrm{nl}}^2(a)\big)\alpha^{(j-1,l+1)} + \dfrac{lp}{m}\alpha^{(j,l-1)} \\ \qquad - j\big[eE\cos\omega t - mx\big(\omega_0^2 + \zeta(a+b)\omega_{\mathrm{nl}}^2(a+b)\big)\big]\alpha^{(j-1,l)}. \end{cases} \qquad (1.14)$$

Here $j,l,M \in \mathbb{N}$, $j+l = \overline{3,M}$, and we have designated $\omega_0 = \sqrt{k/m}$, $\omega_{\mathrm{nl}}(u) = \sqrt{|\tilde{\varkappa}u|/m}$, $\zeta(u) = \mathrm{sign}(\tilde{\varkappa}u)$.

We shall call the system of equations (1.12)–(1.14) *the Hamilton–Ehrenfest* system of order $M$ ($M$ is the order of the greatest moment taken into account) that is associated with the Hartree-type equation (0.1).

Let us consider the Hamilton–Ehrenfest system (1.12)–(1.14) as an abstract system of ordinary differential equations with arbitrary initial conditions. It is obvious that not all solutions of the Hamilton–Ehrenfest system (1.12)–(1.14) can be obtained by averaging the corresponding operators over the solutions of the Hartree-type equation (0.1). For example, the average values should satisfy the Schrödinger indeterminacy relation

$$\sigma_{pp}\sigma_{xx} - \sigma_{xp}^2 \geqslant \frac{\hbar^2}{4} \qquad (1.15)$$

for the second-order moments (indeterminacy relations for higher-order moments are given in [19, 20]), while the Hamilton–Ehrenfest system admits the trivial solutions $p = 0$, $x = 0, \alpha^{(j,l)} = 0, j + l = \overline{2, M}$. The quantity on the left-hand side of relation (1.15) is the integral of motion for the Hamilton–Ehrenfest system (1.13) (see [21]). Hence, it suffices that the indeterminacy relation be fulfilled for the time zero.

The indeterminacy relations will be fulfilled automatically if we choose for the Hamilton–Ehrenfest system (1.12)–(1.13) the following initial conditions:

$$
\begin{aligned}
&p|_{t=s} = p_0 = p_\psi(\hbar) && x|_{t=s} = x_0 = x_\psi(\hbar) \\
&\sigma_{pp}|_{t=s} = \alpha_\psi^{(2,0)}(\hbar) && \sigma_{xp}|_{t=s} = \alpha_\psi^{(1,1)}(\hbar) && \sigma_{xx}|_{t=s} = \alpha_\psi^{(0,2)}(\hbar) \\
&\alpha^{(j,l)}|_{t=s} = \alpha_\psi^{(j,l)} && j, l, M \in \mathbb{N} \quad j + l = \overline{3, M}
\end{aligned}
\tag{1.16}
$$

where $\psi(x, \hbar)$ is the initial condition (1.8) for equation (0.1).

Let us designate

$$
\tilde{\Omega} = \sqrt{\left|\omega_0^2 + \zeta(a+b)\omega_{\mathrm{nl}}^2(a+b)\right|} \qquad \Omega = \sqrt{\left|\omega_0^2 + \zeta(a)\omega_{\mathrm{nl}}^2(a)\right|}.
\tag{1.17}
$$

The Hamilton–Ehrenfest system of equations (1.12)–(1.14) breaks into $M$ recurrent systems: a system of equations for the first non-centred (initial) moments $p_\Psi(t)$, $x_\Psi(t)$ and a system of equations for centred moments $\alpha^{(j,l)}$ of order $n$, $n = j + l, n = \overline{2, M}$. The system of order $n$ does not depend on the moments of order above $n$.

The general solutions of systems (1.12) and (1.13), depending on the sign on the expressions under the module sign in (1.17), are different in structure. Here we shall restrict ourselves to the case where the inequalities $\tilde{\Omega}^2 = \omega_0^2 + \zeta(a + b)\omega_{\mathrm{nl}}^2(a + b) > 0$ and $\Omega^2 = \omega_0^2 + \zeta(a)\omega_{\mathrm{nl}}^2(a) > 0$ are satisfied simultaneously. Thus, the general solution of systems (1.12) and (1.13) has the form

$$
\begin{aligned}
X(t) &= C_1 \sin \tilde{\Omega}t + C_2 \cos \tilde{\Omega}t + \frac{eE}{m(\tilde{\Omega}^2 - \omega^2)} \cos \omega t \\
P(t) &= m\tilde{\Omega}C_1 \cos \tilde{\Omega}t - m\tilde{\Omega}C_2 \sin \tilde{\Omega}t - \frac{eE\omega}{(\tilde{\Omega}^2 - \omega^2)} \sin \omega t
\end{aligned}
\tag{1.18}
$$

$$
\begin{aligned}
&\sigma_{xx}(t) = C_3 \sin 2\Omega t + C_4 \cos 2\Omega t + C_5 && \sigma_{xp}(t) = m\Omega C_3 \cos 2\Omega t - m\Omega C_4 \sin 2\Omega t \\
&\sigma_{pp}(t) = -m^2\Omega^2 C_3 \sin 2\Omega t - m^2\Omega^2 C_4 \cos 2\Omega t + m^2\Omega^2 C_5.
\end{aligned}
\tag{1.19}
$$

Here, $C_l, l = \overline{1, 5}$, are arbitrary constants.

Let us designate by $\mathfrak{g} = \mathfrak{g}(t, \mathfrak{C}) \in \mathbb{R}^5$ a trajectory in an extended phase space. Here

$$
\begin{aligned}
\mathfrak{g}(t, \mathfrak{C}) &= (P(t, \mathfrak{C}), X(t, \mathfrak{C}), \sigma_{pp}(t, \mathfrak{C}), \sigma_{px}(t, \mathfrak{C}), \sigma_{xx}(t, \mathfrak{C}))^\mathsf{T} \\
\mathfrak{C} &= (C_1, C_2, C_3, C_4, C_5)^\mathsf{T}
\end{aligned}
\tag{1.20}
$$

is the general solution of the Hamilton–Ehrenfest system (1.12), (1.13) and $\hat{\mathfrak{g}}$ is a column of operators:

$$
\hat{\mathfrak{g}} = \left(\hat{p}, \hat{x}, (\Delta\hat{p})^2, \tfrac{1}{2}(\Delta\hat{p}\Delta x - \Delta x\Delta\hat{p}), (\Delta x)^2\right)^\mathsf{T}.
\tag{1.21}
$$

Here, $B^\mathsf{T}$ is the transpose of the matrix $B$. The systems of equations (1.12), (1.13) can be written as

$$
\dot{\mathfrak{g}} = \mathfrak{A}\mathfrak{g} + \mathfrak{a}(t) \qquad \mathfrak{g}|_{t=s} = \mathfrak{g}_0 \qquad \mathfrak{a}(t) = (eE \cos \omega t, 0, 0, 0, 0)^\mathsf{T}
\tag{1.22}
$$

where

$$
\mathfrak{A} = \begin{pmatrix}
0 & -m\tilde{\Omega}^2 & 0 & 0 & 0 \\
\dfrac{1}{m} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -2m\Omega^2 & 0 \\
0 & 0 & \dfrac{1}{m} & 0 & -m\Omega^2 \\
0 & 0 & 0 & \dfrac{2}{m} & 0
\end{pmatrix}.
$$

**Theorem 1.1.** *Let $\Psi(x, t)$ be a particular solution of the Hartree-type equation (0.1) with the initial condition $\Psi(x, t)|_{t=0} = \psi(x)$. Let us determine the constants $\mathfrak{C}(\Psi(t))$ from the condition*

$$
\mathfrak{g}(t, \mathfrak{C}) = \langle \Psi(t) | \hat{\mathfrak{g}} | \Psi(t) \rangle \tag{1.23}
$$

*and the constants $\mathfrak{C}(\psi)$ from the condition*

$$
\mathfrak{g}(0, \mathfrak{C}) = \langle \psi | \hat{\mathfrak{g}} | \psi \rangle. \tag{1.24}
$$

*Then we have $\mathfrak{C}(\Psi(t)) = \mathfrak{C}(\psi)$.*

**Proof.** By construction, the vector

$$
\mathfrak{g}(t) = \langle \Psi(t) | \hat{\mathfrak{g}} | \Psi(t) \rangle = \mathfrak{g}(t, \mathfrak{C}(\Psi(t))) \tag{1.25}
$$

is a particular solution of the system of equations (1.12), (1.13) and at the time $t = 0$ it coincides with $\mathfrak{g}(t, \mathfrak{C}(\psi))$. By virtue of the uniqueness of the solution of the Cauchy problem for system (1.12), (1.13), the relation

$$
\mathfrak{g}(t, \mathfrak{C}(\psi)) = \mathfrak{g}(t, \mathfrak{C}(\Psi(t))) \tag{1.26}
$$

is fulfilled, and thus the theorem is proved. $\qquad\square$

From equations (1.16) and (1.24) it follows that

$$
\mathfrak{C}(\psi) = \left( \frac{p_0}{m\tilde{\Omega}}, x_0 - \frac{eE}{m(\tilde{\Omega}^2 - \omega^2)}, \frac{\alpha_\psi^{(1,1)}(\hbar)}{m\Omega}, \frac{1}{2}\left( \alpha_\psi^{(0,2)}(\hbar) - \frac{\alpha_\psi^{(2,0)}(\hbar)}{m^2\Omega^2} \right), \right.
$$
$$
\left. \frac{1}{2}\left( \alpha_\psi^{(0,2)}(\hbar) + \frac{\alpha_\psi^{(2,0)}(\hbar)}{m^2\Omega^2} \right) \right)^{\mathsf{T}}. \tag{1.27}
$$

## 2. Associated Schrödinger equation

Let us take the Taylor series in $\Delta x = x - x_\Psi(t, \hbar)$, $\Delta y = y - x_\Psi(t, \hbar)$ and $\Delta \hat{p} = \hat{p} - p_\Psi(t, \hbar)$ for the operators entering equation (0.1). Then equation (0.1) takes the form

$$
\left\{ -\mathrm{i}\hbar \partial_t + \mathfrak{H}(t, \Psi(t)) + \langle \mathfrak{H}_z(t, \Psi(t)), \Delta\hat{z} \rangle + \frac{1}{2}\langle \Delta\hat{z}, \mathfrak{H}_{zz}(t, \Psi(t))\Delta\hat{z} \rangle \right\} \Psi = 0
$$

$$
\mathfrak{H}(t, \Psi(t)) = \frac{p_\Psi^2(t, \hbar)}{2m} + \frac{kx_\Psi^2(t, \hbar)}{2} - eEx_\Psi(t, \hbar)\cos\omega t
$$
$$
+ \frac{\tilde{\varkappa}}{2}c\alpha_\Psi^{(0,2)}(t, \hbar) + \frac{\tilde{\varkappa}}{2}(a + 2b + c)x_\Psi^2(t, \hbar)
$$

$$\mathfrak{H}_z(t, \Psi(t)) = \begin{pmatrix} \dfrac{1}{m} p_\Psi(t, \hbar) \\ m\tilde{\Omega}^2 x_\Psi(t, \hbar) - eE\cos\omega t \end{pmatrix} \qquad \Delta\hat{z} = \begin{pmatrix} \Delta\hat{p} \\ \Delta x \end{pmatrix}$$

$$\mathfrak{H}_{zz}(t, \Psi(t)) = \begin{pmatrix} \dfrac{1}{m} & 0 \\ 0 & m\Omega^2 \end{pmatrix} \qquad z = \begin{pmatrix} p \\ x \end{pmatrix}.$$

$$(2.1)$$

Here $\langle \cdots \rangle$ denotes an Euclidean scalar product of vectors. Let us associate the nonlinear equation (2.1) with the linear equation that is obtained from (2.1) by substituting the corresponding solutions of the Hamilton–Ehrenfest system (1.12), (1.13) for the average values of the operators of coordinates, momenta and second-order centred moments. As a result, we obtain the following equation:

$$\left\{ -i\hbar\partial_t + \mathfrak{H}(t, \hbar, \mathfrak{g}(t, \mathfrak{C})) + \frac{1}{2}\langle \mathfrak{H}_z(t, \mathfrak{g}(t, \mathfrak{C})), \Delta\hat{z} \rangle + \langle \Delta\hat{z}, \mathfrak{H}_{zz}(t, \mathfrak{g}(t, \mathfrak{C}))\Delta\hat{z} \rangle \right\} \Phi = 0$$

$$\mathfrak{H}(t, \hbar, \mathfrak{g}(t, \mathfrak{C})) = \frac{P^2(t, \mathfrak{C})}{2m} + \frac{kX^2(t, \mathfrak{C})}{2} - eEX(t, \mathfrak{C})\cos\omega t$$

$$+ \frac{\tilde{\varkappa}}{2}c\sigma_{xx}(t, \mathfrak{C}) + \frac{\tilde{\varkappa}}{2}(a + 2b + c)X^2(t, \mathfrak{C}) \qquad (2.2)$$

$$\mathfrak{H}_z(t, \mathfrak{g}(t, \mathfrak{C})) = \begin{pmatrix} \dfrac{1}{m} P(t, \mathfrak{C}) \\ m\tilde{\Omega}^2 X(t, \mathfrak{C}) - eE\cos\omega t \end{pmatrix} \qquad \mathfrak{H}_{zz}(t, \mathfrak{g}(t, \mathfrak{C})) = \begin{pmatrix} \dfrac{1}{m} & 0 \\ 0 & m\Omega^2 \end{pmatrix}.$$

We shall call equation (2.2) the *associated linear Schrödinger equation*.

Equation (2.2) is a Schrödinger equation with a quadratic Hamiltonian. It is well known that this type of equation (see, e.g., [22, 23]) admits solutions in the form of Gaussian wave packets. Let us construct Fock's basis of the solutions of equation (2.2) in the form accepted in the complex germ theory [10, 11]. By direct check we verify that the function

$$\Phi_0^{(0)}(x, t, \mathfrak{g}(t, \mathfrak{C})) = N_\hbar \left( \frac{1}{C(t)} \right)^{1/2} \exp\left\{ \frac{i}{\hbar}\left( S(t, \hbar, \mathfrak{g}(t, \mathfrak{C})) + P(t, \mathfrak{C})\Delta x + \frac{1}{2}\frac{B(t)}{C(t)}\Delta x^2 \right) \right\}$$

$$(2.3)$$

is a solution of equation (2.2), where

$$S(t, \hbar, \mathfrak{g}(t, \mathfrak{C})) = \int_0^t \left( P(t, \mathfrak{C})\dot{X}(t, \mathfrak{C}) - \mathfrak{H}(t, \hbar, \mathfrak{g}(t, \mathfrak{C})) \right) dt. \qquad (2.4)$$

Here, $B(t)$ and $C(t)$ designate, respectively, the 'momentum' and the 'coordinate' parts of the solution of the system of equations in variations

$$\dot{a} = J\mathfrak{H}_{zz}(t, \mathfrak{g}(t, \mathfrak{C}))a \qquad a(t) = \left( B(t)C(t) \right)^\mathsf{T} \qquad (2.5)$$

corresponding to equation (2.2). Let us set up the Floquet problem for the system of equations (2.5)

$$a(t + T) = e^{i\Omega T} a(t). \qquad (2.6)$$

The solution of the Floquet problem (2.5), (2.6) normalized by the condition

$$\{a(t), a^*(t)\} = 2i \qquad \{a_1, a_2\} = \langle a_1, J^\mathsf{T} a_2 \rangle \qquad (2.7)$$

where $J$ is the unit symplectic matrix, has the form

$$a(t) = \frac{e^{i\Omega t}}{\sqrt{m\Omega}}(im\Omega, 1)^\mathsf{T}. \qquad (2.8)$$

From the normalization condition $\left\| \Phi_0^{(0)}(x, t, \mathfrak{g}(t, \mathfrak{C})) \right\|^2 = 1$ we obtain $N_\hbar = (1/\pi\hbar)^{1/4}$.

Let us designate

$$\hat{a}(t) = N_a(C(t)\Delta\hat{p} - B(t)\Delta x). \tag{2.9}$$

If $C(t)$ and $B(t)$ are solutions of the system of equations (2.5), then the operator $\hat{a}(t)$ commutes with the operator of the associated equation (2.2). Thus, the functions

$$\Phi_n^{(0)}(x, t, \mathfrak{g}(t, \mathfrak{C})) = \frac{1}{\sqrt{n!}}(\hat{a}^+(t))^n \Phi_0^{(0)}(x, t, \mathfrak{g}(t, \mathfrak{C})) \qquad n = \overline{0, \infty}$$

are also solutions of the Schrödinger equation (2.2). Commuting the operators $\hat{a}^+(t)$ with the operator of multiplication by the function $\Phi_0^{(0)}(x, t, \mathfrak{g}(t, \mathfrak{C}))$, we obtain the following presentation for the Fock basis of the solutions of the associated linear equation (2.2)

$$\Phi_n^{(0)}(x, t, \mathfrak{g}(t, \mathfrak{C})) = \frac{1}{\sqrt{n!}}N_a^n \Phi_0^{(0)}(x, t, \mathfrak{g}(t, \mathfrak{C}))(-\mathrm{i})^n[C^*(t)]^n \left[\hbar\frac{\partial}{\partial x} - \frac{2m}{|C(t)|^2}\Delta x\right]^n 1$$

$$= \frac{1}{\sqrt{n!}}N_a^n \Phi_0^{(0)}(x, t, \mathfrak{g}(t, \mathfrak{C}))\mathrm{i}^n[C^*(t)]^n \left(\frac{\sqrt{\hbar m}}{|C(t)|}\right)^n H_n\left(\Delta x \frac{\sqrt{m}}{|C(t)|\sqrt{\hbar}}\right)$$

where $H_n(\xi)$ are Hermitean polynomials. Finding $N_a = 1/\sqrt{2\hbar}$ from the condition

$$[\hat{a}(t), \hat{a}^+(t)] = 1$$

and presenting the coordinate part of the solution of the system of equations in variations in the form

$$C(t) = \frac{1}{\sqrt{m\Omega}}\exp\{\mathrm{i}\Omega t\}$$

we obtain

$$\Phi_n^{(0)}(x, t, \mathfrak{g}(t, \mathfrak{C})) = \frac{\mathrm{i}^n}{\sqrt{n!}}\exp\{-\mathrm{i}n\Omega t\}\left(\frac{1}{\sqrt{2}}\right)^n H_n\left(\sqrt{\frac{m\Omega}{\hbar}}\Delta x\right)\Phi_0^{(0)}(x, t, \mathfrak{g}(t, \mathfrak{C})). \tag{2.10}$$

Using properties of the Hermite polynomials, we have

$$\alpha_{\Phi_n^{(0)}}^{(0,2)}(t, \hbar) = \sigma_{xx}(t, \hbar, \mathfrak{C}) = \frac{1}{2^n n!\sqrt{\pi}}\int_{-\infty}^{\infty}\Delta x^2 \left|\Phi_n^{(0)}(x, t, \mathfrak{g}(t, \mathfrak{C}))\right|^2 \mathrm{d}x = \frac{\hbar(2n+1)}{2m\Omega}. \tag{2.11}$$

We have already mentioned that the functions $\left\{\Phi_n^{(0)}(x, t, \mathfrak{g}(t, \mathfrak{C}))\right\}_{n=0}^{\infty}$, while being solutions of the associated Schrödinger equation (2.2), in the general case (with arbitrary parameters $\mathfrak{C}$) are not solutions of the Hartree-type equation (0.1). The functions (2.10) with fixed $n$ will be solutions of equation (0.1) only if the parameters $\mathfrak{C}$ are specially chosen, namely by the rule (1.27).

Let us put $\psi(x) = \Phi_n^{(0)}(x, 0)$ in (1.27). In view of (2.11), we obtain

$$\mathfrak{C}_n = \mathfrak{C}\left(\Phi_n^{(0)}(0)\right) = \left(\frac{p_0}{m\tilde{\Omega}}, x_0, 0, 0, \frac{\hbar(2n+1)}{2m\Omega}\right)^{\mathsf{T}}. \tag{2.12}$$

**Theorem 2.1.** *For every fixed $n$, $n = \overline{0, \infty}$, the functions*

$$\Psi_n^{(0)}(x, t, \mathfrak{g}(t, \mathfrak{C}_n)) = \frac{\mathrm{i}^n}{\sqrt{n!}}\mathrm{e}^{-\mathrm{i}n\Omega t}\left(\frac{1}{\sqrt{2}}\right)^n H_n\left(\sqrt{\frac{m\Omega}{\hbar}}\Delta x\right)\Psi_0^{(0)}(x, t, \mathfrak{g}(t, \mathfrak{C}_n)) \tag{2.13}$$

$$\Psi_0^{(0)}(x, t, \mathfrak{g}(t, \mathfrak{C}_n)) = \sqrt[4]{\frac{m\Omega}{\pi\hbar}}\, \mathrm{e}^{-\mathrm{i}\Omega t/2} \exp\left\{\frac{\mathrm{i}}{\hbar}(S(t, \hbar, \mathfrak{g}(t, \mathfrak{C}_n)) + P(t, \mathfrak{C}_n)\Delta x) - \frac{m\Omega}{2\hbar}\Delta x^2\right\}$$

(2.14)

*are exact solutions of equation (0.1). Here, $\Delta x = x - X(t, \mathfrak{C}_n)$ and*

$$S(t, \hbar, \mathfrak{g}(t, \mathfrak{C}_n)) = \int_0^t (P(t, \mathfrak{C}_n)\dot{X}(t, \mathfrak{C}_n) - \mathfrak{H}(t, \hbar, \mathfrak{g}(t, \mathfrak{C}_n)))\,\mathrm{d}t.$$

(2.15)

*The functions (2.13) satisfy the initial conditions*

$$\Psi_n^{(0)}(x, t, \mathfrak{g}(t, \mathfrak{C}_n))|_{t=0} = \Phi_n^{(0)}(x, 0).$$

(2.16)

*Let us call the functions (2.13) semiclassical trajectory coherent states for the Hartree-type equation (0.1).*

Proof of this statement is given below for a more general case.

Associate the solutions of the Hamilton–Ehrenfest system with an additional periodicity condition with period $T = 2\pi/\omega$, to obtain

$$\mathfrak{C}_n^T(\psi) = \left(0, \frac{eE}{m(\tilde{\Omega}^2 - \omega^2)}, 0, 0, \frac{\hbar(2n+1)}{2m\Omega}\right)^{\mathsf{T}}$$

(2.17)

and

$$\begin{aligned}
S\left(t, \hbar, \mathfrak{g}\left(t, \mathfrak{C}_n^T\right)\right) &= \int_0^t \left[P\left(t, \mathfrak{C}_n^T\right)\dot{X}\left(t, \mathfrak{C}_n^T\right) - \mathfrak{H}\left(t, \hbar, \mathfrak{g}\left(t, \mathfrak{C}_n^T\right)\right)\right]\mathrm{d}t \\
&= \frac{e^2 E^2}{2m(\tilde{\Omega}^2 - \omega^2)}\left(1 + \frac{\omega^2 - \omega_0^2 - \zeta(a + 2b + c)\omega_{\mathrm{nl}}^2(a + 2b + c)}{2(\tilde{\Omega}^2 - \omega^2)}\right)t \\
&\quad - \frac{\hbar\tilde{\varkappa}c(2n+1)}{4m\Omega}t + \frac{e^2 E^2}{4m\omega(\tilde{\Omega}^2 - \omega^2)} \\
&\quad \times \left(1 + \frac{-\omega^2 - \omega_0^2 - \zeta(a + 2b + c)\omega_{\mathrm{nl}}^2(a + 2b + c)}{2(\tilde{\Omega}^2 - \omega^2)}\right)\sin(2\omega t).
\end{aligned}$$

(2.18)

In this case, the functions (2.13) satisfy the condition

$$\Psi_n^{(0)}\left(x, t + T, \mathfrak{C}_n^T\right) = \mathrm{e}^{-\mathrm{i}\mathcal{E}_n T/\hbar}\Psi_n^{(0)}\left(x, t, \mathfrak{C}_n^T\right)$$

(2.19)

where $\mathcal{E}$ is the quasi-energy. As a result, we obtain for quasi-energy levels and the Aharonov–Anandan phase [24, 25], respectively

$$\begin{aligned}
\mathcal{E}_n &= -\frac{e^2 E^2}{2m(\tilde{\Omega}^2 - \omega^2)} - \frac{e^2 E^2[\omega^2 - \omega_0^2 - \zeta(a + 2b + c)\omega_{\mathrm{nl}}^2(a + 2b + c)]}{4m(\tilde{\Omega}^2 - \omega^2)^2} \\
&\quad + \hbar\left(\Omega + \frac{\tilde{\varkappa}c}{2m\Omega}\right)\left(n + \frac{1}{2}\right)
\end{aligned}$$

(2.20)

$$\gamma_{\mathcal{E}_n} = \frac{1}{\hbar}\frac{T e^2 E^2\omega^2}{2m(\tilde{\Omega}^2 - \omega^2)^2}.$$

(2.21)

## 3. Nonlinear evolution operator

Theorem 2.1 gives a solution of the Cauchy problem (0.1), (1.8) for a special class of initial conditions of the form (2.16). To find the Cauchy problem solution with arbitrary initial

conditions in the class of semiclassically concentrated functions, we construct the evolution operator for the Hartree-type equation (0.1).

The system of functions $\left\{\Phi_n^{(0)}(x, t, \mathfrak{g}(t, \mathfrak{C}))\right\}_{n=0}^{\infty}$ of the form (2.10) constitutes a complete set of solutions of the associated linear Schrödinger equation (2.2) and makes it possible to construct the evolution operator of the Hartree-type equation (0.1). The Green function of the linear equation (the core of the evolution operator) can be expanded over the complete set of solutions of the linear equation:

$$G^{(0)}(x, y, t, s, \mathfrak{g}(t, \mathfrak{C}), \mathfrak{g}(s, \mathfrak{C})) = \sum_{n=0}^{\infty} \Phi_n^{(0)}(x, t, \mathfrak{g}(t, \mathfrak{C}))\left(\Phi_n^{(0)}(y, s, \mathfrak{g}(s, \mathfrak{C}))\right)^*. \tag{3.1}$$

In view of the Meller formula [26]

$$\sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\lambda}{2}\right)^n H_n(x) H_n(y) = \frac{1}{\sqrt{1-\lambda^2}} \exp\left[\frac{2xy\lambda - (x^2 + y^2)\lambda^2}{1-\lambda^2}\right]$$

from (2.10), (3.1) we find

$$G(x, y, t, s, \mathfrak{g}(t, \mathfrak{C}), \mathfrak{g}(s, \mathfrak{C})) = \sqrt{\frac{m\Omega}{2\pi\,i\hbar\,\sin[\Omega(t-s)]}}$$

$$\times \exp\left\{\frac{i}{\hbar}[S(t, \hbar, \mathfrak{g}(t, \mathfrak{C})) - S(s, \hbar, \mathfrak{g}(s, \mathfrak{C})) + P(t, \mathfrak{C})\Delta x - P(s, \mathfrak{C})\Delta y]\right\}$$

$$\times \exp\left\{-\frac{im\Omega}{2\hbar}\left(\frac{2\Delta x \Delta y - (\Delta x^2 + \Delta y^2)\cos[\Omega(t-s)]}{\sin[\Omega(t-s)]}\right)\right\}.$$

**Theorem 3.1.** *Let the operator $\widehat{U}_\varkappa(t, s, \cdot)$ be defined by the relation*

$$\widehat{U}_\varkappa(t, s, \psi)(x) = \int_{-\infty}^{\infty} G_\varkappa(x, y, t, s, \mathfrak{g}(t, \mathfrak{C}(\psi)), \mathfrak{g}(s, \mathfrak{C}(\psi)))\psi(y)\,\mathrm{d}y \tag{3.2}$$

*where*

$$G_\varkappa(x, y, t, s, \mathfrak{g}(t, \mathfrak{C}(\psi)), \mathfrak{g}(s, \mathfrak{C}(\psi))) = \sqrt{\frac{m\Omega}{2\pi\,i\hbar\,\sin[\Omega(t-s)]}}$$

$$\times \exp\left\{\frac{i}{\hbar}[S(t, \hbar, \mathfrak{g}(t, \mathfrak{C}(\psi))) + P(t, \mathfrak{C}(\psi))\Delta x\right.$$

$$\left. - S(s, \hbar, \mathfrak{g}(s, \mathfrak{C}(\psi))) - (s, \mathfrak{C}(\psi))\Delta y]\right\}$$

$$\times \exp\left\{-\frac{im\Omega}{2\hbar}\left(\frac{2\Delta x \Delta y - (\Delta x^2 + \Delta y^2)\cos[\Omega(t-s)]}{\sin[\Omega(t-s)]}\right)\right\}. \tag{3.3}$$

*Here, $\Delta x = x - X(t, \mathfrak{C}(\psi))$, $\Delta y = y - X(s, \mathfrak{C}(\psi))$, the function $S(t, \hbar, \mathfrak{g}(t, \mathfrak{C}(\psi)))$ is defined by (2.4), and the parameters $\mathfrak{C}(\psi)$ are determined from the equation*

$$\mathfrak{g}(t, \mathfrak{C})|_{t=s} = \mathfrak{g}_0(\psi) = \langle\psi|\hat{\mathfrak{g}}|\psi\rangle. \tag{3.4}$$

*Then the function*

$$\Psi(x, t) = \widehat{U}_\varkappa(t, s, \psi)(x) \tag{3.5}$$

*is an exact solution of the Cauchy problem for the Hartree-type equation (0.1) with the initial condition $\Psi(x, t)|_{t=s} = \psi(x)$, and the operator $\widehat{U}_\varkappa(t, s, \cdot)$ is the evolution operator for the nonlinear Hartree-type equation (0.1).*

The proof of this theorem can be performed by immediate substitution. Details of the calculations are given in appendix A.

**Theorem 3.2.** *Let the operator $\widehat{U}_\varkappa^{-1}(t, s, \cdot)$ be defined by the relation*

$$\widehat{U}_\varkappa^{-1}(t, s, \psi)(x) = \int_{-\infty}^{\infty} G_\varkappa^{-1}(x, y, t, s, \mathfrak{g}(t, (\psi)), \mathfrak{g}(s, \mathfrak{C}(\psi)))\psi(y)\,\mathrm{d}y$$

$$= \int_{-\infty}^{\infty} G_\varkappa(x, y, s, t, \mathfrak{g}(s, \mathfrak{C}(\psi)), \mathfrak{g}(t, (\psi)))\psi(y)\,\mathrm{d}y. \tag{3.6}$$

*Here, the function $G_\varkappa(x, y, s, t, \mathfrak{g}(s, \mathfrak{C}(\psi)), \mathfrak{g}(t, (\psi)))$ is defined by relation (3.2) in which the variable t should be replaced by s and the variable s by t; the constants $\mathfrak{C}$ are determined from the equation*

$$\mathfrak{g}(s, \mathfrak{C})|_{s=t} = \mathfrak{g}_0(\psi) = \langle \psi|\widehat{\mathfrak{g}}|\psi\rangle. \tag{3.7}$$

*The operator $\widehat{U}_\varkappa^{-1}(t, s, \cdot)$ (3.6) is then the left inverse one to the operator $\widehat{U}_\varkappa(t, s, \cdot)$ (3.2), that is,*

$$\widehat{U}_\varkappa^{-1}(t, s, \widehat{U}_\varkappa(t, s, \psi))(x) = \psi(x) \qquad \psi \in \mathcal{P}_\hbar^0. \tag{3.8}$$

Proof of this theorem can be performed by immediate substitution. Details of the calculations are given in appendix B.

**Corollary 2.1.** *If the function $\Psi(x, t)$ is a particular solution of the Hartree-type equation (0.1), then*

$$\widehat{U}_\varkappa\big(t, s, \widehat{U}_\varkappa^{-1}(t, s, \Psi(t))\big)(x) = \Psi(x, t). \tag{3.9}$$

**Proof.** Let us designate $\psi(x) = \Psi(x, t)|_{t=s}$. Then by virtue of theorem 3.1, we have $\Psi(x, t) = \widehat{U}_\varkappa(t, s, \psi)(x)$. Therefore, the left-hand side of relation (3.9) can be presented in the form

$$\widehat{U}_\varkappa\big(t, s, \widehat{U}_\varkappa^{-1}(t, s, \Psi(t))\big)(x) = \widehat{U}_\varkappa\big[t, s, \widehat{U}_\varkappa^{-1}(t, s, \widehat{U}_\varkappa(t, s, \psi))\big](x)$$

$$= \widehat{U}_\varkappa(t, s, \psi)(x) = \Psi(x, t).$$

Here, we used formula (3.8). So, the statement is proved. □

## 4. Symmetry operators for the Hartree-type equation

The solution of the Cauchy problem for the Hartree-type equation (0.1) in the class of semiclassically concentrated functions $\mathcal{P}_\hbar^0$ and the explicit form of the evolution operator (3.2) allow one, in turn, to construct in explicit form the general expressions for the basic constructions of symmetry analysis [1–6]. These constructions are: symmetry operators, a one-parametric family of symmetry operators and generators of this family (symmetries of equation (0.1)).

Actually, let $\hat{a}$ be some operator acting in $\mathcal{P}_\hbar^0$, $\big(\hat{a} : \mathcal{P}_\hbar^0 \to \mathcal{P}_\hbar^0\big)$ and $\Psi(x, t)$ is any function from the class $\mathcal{P}_\hbar^t$. Let us define the operator $\widehat{A}(\cdot)$ by the relation

$$\Phi(x, t) = \widehat{A}(\Psi(t))(x) = \widehat{U}_\varkappa\big(t, \hat{a}\widehat{U}_\varkappa^{-1}(t, \Psi(t))\big)(x) \tag{4.1}$$

where $\widehat{U}_\varkappa(t, \cdot) = \widehat{U}_\varkappa(t, 0, \cdot)$.

**Theorem 4.1.** *If the function* $\Psi(x, t)$ *is a solution of the Hartree-type equation (0.1), then* $\Phi(x, t)$ *(4.1) is also the solution of equation (0.1).*

Proof of this theorem immediately follows from theorems 3.1 and 3.2.

Thus, the operator $\widehat{\mathsf{A}}(\cdot)$, determined by relation (4.1), is the symmetry operator for equation (0.1).

Now let the operator $\hat{\mathsf{b}}$ and its operator exponent $\exp(\alpha\hat{\mathsf{b}})$ act in the class $\mathcal{P}_\hbar^0$; that is, $\hat{\mathsf{b}} : \mathcal{P}_\hbar^0 \to \mathcal{P}_\hbar^0$ and $\exp(\alpha\hat{\mathsf{b}}) : \mathcal{P}_\hbar^0 \to \mathcal{P}_\hbar^0$, where $\alpha$ is a real parameter. For an arbitrary function $\Psi(x, t) \in \mathcal{P}_\hbar^t$, let us define a one-parametric family of operators $\widehat{\mathsf{B}}(\alpha, \cdot)$ by the relation

$$\widehat{\mathsf{B}}(\alpha, \Psi(t))(x) = \widehat{U}_\varkappa\big(t, \exp\{\alpha\hat{\mathsf{b}}\}\widehat{U}_\varkappa^{-1}(t, \Psi(t))\big)(x). \tag{4.2}$$

By analogy with the above constructions, the operators $\widehat{\mathsf{B}}(\alpha, \cdot)$ constitute the one-parametric family of symmetry operators of equation (0.1).

It is easy to verify that for an arbitrary function $\Psi(x, t) \in \mathcal{P}_\hbar^t$ the group property is valid:

$$\widehat{\mathsf{B}}(\alpha + \beta, \Psi(t))(x) = \widehat{\mathsf{B}}(\alpha, \widehat{\mathsf{B}}(\beta, \Psi(t)))(x). \tag{4.3}$$

Differentiating relation (4.2) with respect to the parameter $\alpha$ at the point $\alpha = 0$, we obtain

$$\widehat{\mathsf{C}}(\Psi(t))(x) = \frac{\mathrm{d}}{\mathrm{d}\alpha}\widehat{\mathsf{B}}(\alpha, \Psi(t))(x)\bigg|_{\alpha=0} = \frac{\mathrm{d}}{\mathrm{d}\alpha}\widehat{U}_\varkappa\big(t, \exp\{\alpha\hat{\mathsf{b}}\}\widehat{U}_\varkappa^{-1}(t, \Psi(t))\big)(x)\bigg|_{\alpha=0}. \tag{4.4}$$

The operator $\widehat{\mathsf{C}}(\cdot)$, defined by relation (4.4), is the generator of the one-parameter family of symmetry operators (4.3).

Note that the operator $\widehat{\mathsf{C}}(\cdot)$ is not a symmetry operator of equation (0.1). This is due to the fact that the parameters $\mathfrak{C}$ that determine the evolution operator $\widehat{U}_\varkappa(t, \cdot)$ in relation (4.4) depend on $\alpha$. Actually, the quantities $\mathfrak{C}$ are determined from equation (3.4):

$$\mathfrak{g}(t, \mathfrak{C})|_{t=0} = \langle\exp\{\alpha\hat{\mathsf{b}}\}\phi|\hat{\mathfrak{g}}|\exp\{\alpha\hat{\mathsf{b}}\}\phi\rangle \qquad \phi(x) = \widehat{U}_\varkappa^{-1}(t, \Psi(t))(x) \qquad \Psi(x, t) \in \mathcal{P}_\hbar^t$$

which contains the parameter $\alpha$ in explicit form. Thus, expression (4.4) will contain the derivative of the evolution operator $\widehat{U}_\varkappa(t, \cdot)$ with respect to the parameters $\mathfrak{C}$. This derivative is an operator other than the evolution operator.

**Example.** Let us substitute into relation (4.1), instead of the operators $\hat{\mathsf{a}}$, the operators $\hat{a}^+(t)$ and $\hat{a}(t)$ of the form (2.9) for $t = 0$. Here,

$$\hat{a}(0) = \frac{1}{\sqrt{2\hbar m\Omega}}[\Delta\hat{p}_0 - im\Omega\Delta x_0] \qquad \hat{a}^+(0) = \frac{1}{\sqrt{2\hbar m\Omega}}[\Delta\hat{p}_0 + im\Omega\Delta x_0]$$

where $\Delta\hat{p}_0 = -i\hbar\partial_x - p_0$ and $\Delta x_0 = x - x_0$. Then the operators $\widehat{A}^{(\pm)}(\cdot)$ determined by the relations

$$\widehat{A}^{(+)}(\Psi(t))(x) = \widehat{U}_\varkappa\big(t, \hat{a}^+(0)\widehat{U}_\varkappa^{-1}(t, \Psi(t))\big)(x)$$
$$\widehat{A}^{(-)}(\Psi(t))(x) = \widehat{U}_\varkappa\big(t, \hat{a}(0)\widehat{U}_\varkappa^{-1}(t, \Psi(t))\big)(x)$$

where $\Psi(x, t) \in \mathcal{P}_\hbar^t$, are the symmetry operators of equation (0.1). For these operators, we have in particular

$$\widehat{A}^{(+)}\big(\Psi_n^{(0)}(t, \mathfrak{g}(t, \mathfrak{C}_n))\big)(x) = \sqrt{n+1}\Psi_{n+1}^{(0)}(x, t, \mathfrak{g}(t, \mathfrak{C}_{n+1}))$$
$$\widehat{A}^{(-)}\big(\Psi_n^{(0)}(t, \mathfrak{g}(t, \mathfrak{C}_n))\big)(x) = \sqrt{n}\Psi_{n-1}^{(0)}(x, t, \mathfrak{g}(t, \mathfrak{C}_{n-1})).$$

Here, $\Psi_n^{(0)}(x, t, \mathfrak{g}(t, \mathfrak{C}_n))$ are semiclassical trajectory coherent states of the form (2.13), where constants $\mathfrak{C}_n$ are defined by relation (2.12). Therefore, the operators $\widehat{A}^{(+)}(\cdot)$ are nonlinear

analogues of 'creation–annihilation' operators. With the help of the operators $\widehat{A}^{(\pm)}(\cdot)$, relations (2.13) and (2.12) can be presented in the form

$$\Psi_n^{(0)}(x, t, \mathfrak{g}(t, \mathfrak{C}_n)) = \frac{1}{\sqrt{n!}}(\widehat{A}^{(+)}(\cdot))^n \Psi_0^{(0)}(x, t, \mathfrak{g}(t, \mathfrak{C}_0)). \tag{4.5}$$

Let us define the one-parametric family of shift operators $\widehat{D}(\alpha, \cdot)$ for the functions $\Psi(x, t) \in \mathcal{P}_\hbar^t$ by the relation

$$\widehat{D}(\alpha, \Psi(t))(x) = \widehat{U}_\varkappa\big(t, \hat{\mathcal{D}}_0(\alpha)\widehat{U}_\varkappa^{-1}(t, \Psi(t))\big)(x) \tag{4.6}$$

where

$$\widehat{\mathcal{D}}_0(\alpha) = \exp\{\alpha\hat{a}^+(0) - \alpha^*\hat{a}(0)\} \qquad \Psi(x, t) \in \mathcal{P}_\hbar^t \qquad \alpha \in \mathbb{C}. \tag{4.7}$$

The operators $\widehat{D}(\alpha, \cdot)$ are the symmetry operators for equation (0.1), and the functions

$$\Psi_\alpha(x, t, \mathfrak{g}(t, \mathfrak{C}_\alpha)) = \widehat{D}\big(\alpha, \Psi_0^{(0)}(t, \mathfrak{g}(t, \mathfrak{C}_0))\big)(x) \tag{4.8}$$

are the solutions of equation (0.1) for arbitrary complex values of $\alpha$. Here, the parameters $\mathfrak{C}_\alpha$ are defined by equation (3.4), which for this case has the form

$$\mathfrak{g}(t, \mathfrak{C})|_{t=0} = \big\langle \widehat{\mathcal{D}}_0(\alpha)\Psi_0^{(0)}(0)\big|\hat{\mathfrak{g}}\big|\widehat{\mathcal{D}}_0(\alpha)\Psi_0^{(0)}(0)\big\rangle. \tag{4.9}$$

Let us write the operator $\widehat{\mathcal{D}}_0(\alpha)$ as

$$\widehat{\mathcal{D}}_0(\alpha) = \exp\{\beta\Delta\hat{p}_0 + \gamma\Delta x_0\} = \exp\left\{-\frac{i\hbar}{2}\beta\gamma\right\}\exp\{\gamma\Delta x_0\}\exp\{\beta\Delta\hat{p}_0\} \tag{4.10}$$

where $\beta = [\alpha - \alpha^*]/\sqrt{2\hbar m\Omega}$, $\gamma = i[\alpha + \alpha^*]\sqrt{m\Omega/2\hbar}$. Therefore,

$$\Psi_\alpha(x, 0, \mathfrak{g}(0, \mathfrak{C}_\alpha)) = \widehat{\mathcal{D}}_0(\alpha)\Psi_0^{(0)}(x, 0, \mathfrak{g}(0, \mathfrak{C}_0))$$

$$= \exp\left\{-\frac{i\hbar}{2}\beta\gamma\right\}\exp\{\gamma\Delta x_0 - p_0\beta\}\Psi_0^{(0)}(x - i\hbar\beta, 0, \mathfrak{g}(0, \mathfrak{C}_\alpha)). \tag{4.11}$$

$$\Psi_0^{(0)}(x - i\hbar\beta, 0, \mathfrak{g}(0, \mathfrak{C}_\alpha)) = \Psi_0^{(0)}(x, 0, \mathfrak{g}(0, \mathfrak{C}_\alpha))$$

$$\times \exp\left\{\frac{i}{\hbar}\left[-i\hbar p_0\beta - i\hbar Q(0)\Delta x_0\beta - \frac{\hbar^2}{2}Q(0)\beta^2\right]\right\}.$$

Note that $Q(0) = im\Omega$ and, consequently, $\gamma + Q(0)\beta = 2i\alpha\sqrt{m\Omega/2\hbar}$. Similar manipulations yield

$$-\frac{i\hbar}{2}[\beta\gamma + Q(0)\beta^2] = \sqrt{\frac{\hbar m\Omega}{2}}\alpha\beta = \frac{1}{2}[\alpha^2 - |\alpha|^2]. \tag{4.12}$$

Substituting (4.12) into (4.11), we obtain

$$\Psi_\alpha(x, 0, \mathfrak{g}(0, \mathfrak{C}_\alpha)) = \Psi_0^{(0)}(x, 0, \mathfrak{g}(0, \mathfrak{C}_0))\exp\left\{-\frac{|\alpha|^2}{2} + i\sqrt{\frac{2m\Omega}{\hbar}}\alpha\Delta x_0 + \frac{\alpha^2}{2}\right\}$$

$$= \sqrt[4]{\frac{m\Omega}{\pi\hbar}}\exp\left\{-\frac{|\alpha|^2}{2} + \frac{\alpha^2}{2} + \frac{i}{\hbar}(p_0 + \sqrt{2m\Omega\hbar}\alpha)\Delta x_0 - \frac{m\Omega}{2\hbar}\Delta x_0^2\right\}. \tag{4.13}$$

Denoting $\alpha_1 = \operatorname{Re}\alpha$ and $\alpha_2 = \operatorname{Im}\alpha$, we can write

$$|\Psi_\alpha(x, 0, \mathfrak{g}(0, \mathfrak{C}_\alpha))|^2 = \sqrt{\frac{m\Omega}{\pi\hbar}}\exp\left\{-\frac{m\Omega}{\hbar}\left(\Delta x_0 + \sqrt{\frac{2\hbar}{m\Omega}}\alpha_2\right)^2\right\}. \tag{4.14}$$

From equation (4.9), in view of (4.14), we obtain

$$\mathfrak{C}_\alpha = \left( \frac{1}{m\widetilde{\Omega}} p_\alpha, x_\alpha, 0, 0, \frac{\hbar}{2m\Omega} \right)^\mathsf{T} \tag{4.15}$$

where $p_\alpha = p_0 + \alpha_1 \sqrt{2m\Omega\hbar}$, $x_\alpha = x_0 - \alpha_2 \sqrt{2\hbar/m\Omega}$.

As a result, we find an explicit form of the functions $\Psi_\alpha(x, t, \mathfrak{g}(t, \mathfrak{C}_\alpha))$ for the one-parametric family (with the parameter $\alpha$) of solutions for the nonlinear Hartree-type equation (0.1). Thereby, on direct application of the one-parametric family of symmetry operators $\widehat{D}(\alpha, \cdot)$ to the function $\Psi_0(x, t, \mathfrak{g}(t, \mathfrak{C}_0))$ we obtain the explicit expressions

$$\Psi_\alpha(x, t, \mathfrak{g}(t, \mathfrak{C}_\alpha)) = \sqrt[4]{\frac{m\Omega}{\pi\hbar}} \exp\left\{ \frac{i}{\hbar} (S(t, \hbar, \mathfrak{g}(t, \mathfrak{C}_\alpha)) + P(t, \mathfrak{C}_\alpha)\Delta x) - \frac{m\Omega}{2\hbar} \Delta x^2 \right\} \tag{4.16}$$

where $\Delta x = x - X(t, \mathfrak{C}_\alpha)$, and

$$S(t, \hbar, \mathfrak{g}(t, \mathfrak{C}_\alpha)) = \int_0^t (P(t, \mathfrak{C}_\alpha)\dot{X}(t, \mathfrak{C}_\alpha) - \mathfrak{H}(t, \hbar, \mathfrak{g}(t, \mathfrak{C}_\alpha)))\, dt. \tag{4.17}$$

For the operators (4.6), the following multiplication law is valid:

$$\widehat{D}(\alpha, \widehat{D}(\beta, \Psi(t)))(x) = \exp[\alpha\beta^* - \alpha^*\beta]\widehat{D}(\alpha + \beta, \Psi(t))(x). \tag{4.18}$$

The operators $\exp(i\gamma)\widehat{D}(\alpha, \cdot)$, where $\gamma \in \mathbb{R}, \alpha \in \mathbb{C}$, determine the nonlinear analogue of the Heisenberg–Weyl group representation [22, 23]. The function $\Psi_\alpha(x, t, \mathfrak{g}(t, \mathfrak{C}_\alpha))$ (4.16), by virtue of (1.19) and (4.15), minimizes the Schrödinger uncertainty relation (1.15) and hence, they are compressed coherent states. To summarize, we note that the exact expressions constructed in this work for the evolution operator and symmetry analysis constructions for the Hartree-type equation (0.1) can be generalized for the case of a Hartree-type equation in a multidimensional space with smooth coefficients of general form. This can be done based on the results of [8, 9]. However, this generalization will be valid only in the sense of approximation to within $\widehat{O}(\hbar^{(M+1)/2}), \hbar \to 0$, where $M$ is the order of the Hamilton–Ehrenfest system. In particular, this allows one to construct a special kind of approximate symmetry operators (and symmetries) for the above Hartree-type equations, which it is natural to call semiclassical symmetry operators (symmetries).

## Acknowledgments

## Appendix A

**Proof of theorem 3.1.** (1) Let us calculate the limit of the function $\Psi(x, t)$ as $t \to s + 0$. We have

$$\lim_{t \to s+0} \Psi(x, t) = \lim_{t \to s+0} \int_{-\infty}^{\infty} G_\varkappa(x, y, t, s, \mathfrak{g}(t, \mathfrak{C}(\psi)), \mathfrak{g}(s, \mathfrak{C}(\psi)))\psi(y)\, dy$$

$$= \int_{-\infty}^{\infty} dy\, \psi(y) \lim_{t \to s+0} \sqrt{\frac{m\Omega}{2\pi\, i\hbar\, \sin[\Omega(t-s)]}}$$

$$\times \exp\left\{ \frac{i}{\hbar} [S(t, \hbar, \mathfrak{g}(t, \mathfrak{C}(\psi))) - S(s, \hbar, \mathfrak{g}(s, \mathfrak{C}(\psi))) + P(t, \mathfrak{C}(\psi))\Delta x \right.$$

$$\left. - P(s, \mathcal{C}(\psi)) \Delta y \right] \exp\left\{ -\frac{im\Omega}{2\hbar} \left( \frac{2\Delta x \Delta y - (\Delta x^2 + \Delta y^2) \cos[\Omega(t - s)]}{\sin[\Omega(t - s)]} \right) \right\}$$

$$= \int_{-\infty}^{\infty} dy \, \psi(y) \lim_{t \to s+0} \sqrt{\frac{m}{2\pi \, i\hbar(t - s)}} \exp\left\{ \frac{im}{2\hbar} \frac{(\Delta x - \Delta y)^2}{t - s} \right\}$$

$$= \int_{-\infty}^{\infty} dy \, \psi(y) \delta(x - y) = \psi(x).$$

Therefore, $\Psi(x, t)|_{t=s} = \psi(x)$.

(2.) Recall that for the solutions of the Hartree-type equation (0.1), the relation (see (1.26))

$$\mathfrak{g}(t, \mathcal{C}(\psi)) = \langle \Psi(t) | \hat{\mathfrak{g}} | \Psi(t) \rangle \tag{A.1}$$

is valid.

Since it is not yet proved that the functions $\Psi(x, t)$ (3.5) are the solution of equation (0.1), we have to verify by direct check whether relation (A.1) to be used below is valid.

The definition of moments (1.11) involves the norm of the function $\Psi(x, t)$ (3.5). Let us calculate this norm as follows:

$$\|\Psi(t)\|^2 = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \, \psi(y) \psi^*(z) \sqrt{\frac{m\Omega}{2\pi i\hbar \sin[\Omega(s - t)]}} \sqrt{\frac{m\Omega}{2\pi i\hbar \sin[\Omega(t - s)]}}$$

$$\times \exp\left\{ -\frac{i}{\hbar} [S(t, \hbar, \mathfrak{g}(t, \mathcal{C}(\psi))) - S(s, \hbar, \mathfrak{g}(\mathcal{C}(\psi))) \right.$$

$$+ P(t, \mathcal{C}(\psi)) \Delta x - P(s, \mathcal{C}(\psi)) \Delta z \Big\}$$

$$\times \exp\left\{ -\frac{im\Omega}{2\hbar} \left( \frac{2\Delta x \Delta z - (\Delta x^2 + \Delta z^2) \cos[\Omega(t - s)]}{\sin[\Omega(s - t)]} \right) \right\}$$

$$\times \exp\left\{ \frac{i}{\hbar} [S(t, \hbar, \mathfrak{g}(t, \mathcal{C}(\psi))) - S(s, \hbar, \mathfrak{g}(s, \mathcal{C}(\psi))) \right.$$

$$+ P(t, \mathcal{C}(\psi)) \Delta x - P(s, \mathcal{C}(\psi)) \Delta y \Big\}$$

$$\times \exp\left\{ \frac{im\Omega}{2\hbar} \left( \frac{2\Delta x \Delta y - (\Delta x^2 + \Delta y^2) \cos[\Omega(s - t)]}{\sin[\Omega(s - t)]} \right) \right\}$$

$$= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \, \frac{m\Omega}{2\pi\hbar \sin[\Omega(s - t)]} \exp\left\{ \frac{i}{\hbar} [P(s, \mathcal{C}(\psi))(\Delta z - \Delta y)] \right.$$

$$\times \exp\left\{ \frac{im\Omega}{2\hbar} \left( \frac{2(\Delta z - \Delta y)\Delta x - (\Delta z^2 - \Delta y^2) \cos[\Omega(t - s)]}{\sin[\Omega(t - s)]} \right) \right\} \psi(y) \psi^*(z)$$

$$= \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \, \delta(z - y) \exp\left\{ \frac{i}{\hbar} [P(s, \mathcal{C}(\psi))(z - y)] \right\}$$

$$\times \exp\left\{ \frac{im\Omega}{2\hbar} \left( \frac{(\Delta y^2 - \Delta z^2) \cos[\Omega(t - s)]}{\sin[\Omega(t - s)]} \right) \right\} \psi(y) \psi^*(z)$$

$$= \int_{-\infty}^{\infty} dx \, |\psi(x)|^2.$$

Thus, we obtain $\|\Psi(t)\| = \|\psi\|$.

Let us show that

$$x_\Psi(t, \hbar) = X(t, \mathcal{C}(\psi)). \tag{A.2}$$

Calculate the non-centred moment of the first order (A.2). Using the explicit form of the functions $\Psi(x, t)$, we find

$$
\begin{aligned}
x_\Psi(t, \hbar) &= \frac{1}{\|\Psi(t)\|^2} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \, x \sqrt{\frac{m\Omega}{2\pi i\hbar \sin[\Omega(t-s)]}} \sqrt{-\frac{m\Omega}{2\pi i\hbar \sin[\Omega(t-s)]}} \\
&\quad \times \exp\left\{-\frac{i}{\hbar}[S(t, \hbar, \mathfrak{g}(t, \mathfrak{C}(\psi))) - S(s, \hbar, \mathfrak{g}(s, \mathfrak{C}(\psi)))\right. \\
&\qquad \left. + P(t, \mathfrak{C}(\psi))\Delta x - P(s, \mathfrak{C}(\psi))\Delta z]\right\} \\
&\quad \times \exp\left\{-\frac{im\Omega}{2\hbar}\left(\frac{2\Delta x\Delta z - (\Delta x^2 + \Delta z^2)\cos[\Omega(t-s)]}{\sin[\Omega(s-t)]}\right)\right\} \\
&\quad \times \exp\left\{\frac{i}{\hbar}[S(t, \hbar, \mathfrak{g}(t, \mathfrak{C}(\psi))) - S(s, \hbar, \mathfrak{g}(s, \mathfrak{C}(\psi)))\right. \\
&\qquad \left. + P(t, \mathfrak{C}(\psi))\Delta x - P(s, \mathfrak{C}(\psi))\Delta y]\right\} \\
&\quad \times \exp\left\{\frac{im\Omega}{2\hbar}\left(\frac{2\Delta x\Delta y - (\Delta x^2 + \Delta y^2)\cos[\Omega(s-t)]}{\sin[\Omega(s-t)]}\right)\right\} \psi(y)\psi^*(z) \\
&= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \, \frac{m\Omega(X(t, \mathfrak{C}(\psi)) + \Delta x)\psi(y)\psi^*(z)}{2\pi\hbar \sin[\Omega(t-s)]} \\
&\quad \times \exp\left\{\frac{i}{\hbar}[P(s, \mathfrak{C}(\psi))(\Delta z - \Delta y)]\right\} \exp\left\{\frac{im\Omega}{\hbar}\left(\frac{\Delta x(\Delta y - \Delta z)}{\sin[\Omega(s-t)]}\right)\right\} \\
&\quad \times \exp\left\{\frac{im\Omega}{2\hbar}\left(\frac{(\Delta z^2 - \Delta y^2)\cos[\Omega(s-t)]}{\sin[\Omega(s-t)]}\right)\right\} \\
&= X(t, \mathfrak{C}(\psi)) + \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} d\omega \, \frac{\psi(y)\psi^*(z)}{2\pi}\left(\frac{\hbar\omega\sin[\Omega(t-s)]}{m\Omega}\right) \\
&\quad \times \exp\{i\omega(\Delta y - \Delta z)\} \exp\left\{\frac{i}{\hbar}[P(s, \mathfrak{C}(\psi))(\Delta z - \Delta y)]\right\} \\
&\quad \times \exp\left\{\frac{im\Omega}{2\hbar}\frac{(\Delta z^2 - \Delta y^2)\cos[\Omega(s-t)]}{\sin[\Omega(s-t)]}\right\} \\
&= X(t, \mathfrak{C}(\psi)) + \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dy \, \delta'(y-z)\psi(y)\psi^*(z)\frac{\hbar\sin[\Omega(t-s)]}{m\Omega} \\
&\quad \times \exp\left\{\frac{i}{\hbar}[P(s, \mathfrak{C}(\psi))(\Delta z - \Delta y)]\right\} \exp\left\{\frac{im\Omega}{2\hbar}\left(\frac{(\Delta z^2 - \Delta y^2)\cos[\Omega(s-t)]}{\sin[\Omega(s-t)]}\right)\right\} \\
&= X(t, \mathfrak{C}(\psi)) + \frac{1}{2}\int_{-\infty}^{\infty} dz\left\{\frac{\hbar\sin[\Omega(t-s)]}{m\Omega}\exp\left\{\frac{i}{\hbar}P(s, \mathfrak{C}(\psi))(\Delta z - \Delta y)\right\}\right. \\
&\quad \left.\times \exp\left\{\frac{im\Omega}{2\hbar}\frac{(\Delta z^2 - \Delta y^2)\cos[\Omega(s-t)]}{\sin[\Omega(s-t)]}\right\}\psi(y)\psi^*(z)\right\}'_y\bigg|_{y=z} \\
&\quad + \frac{1}{2}\int_{-\infty}^{\infty} dy\left\{\frac{\hbar\sin[\Omega(t-s)]}{m\Omega}\exp\left\{-\frac{i}{\hbar}P(s, \mathfrak{C}(\psi))(\Delta z - \Delta y)\right\}\right. \\
&\quad \left.\times \exp\left\{\frac{im\Omega}{2\hbar}\frac{(\Delta z^2 - \Delta y^2)\cos[\Omega(s-t)]}{\sin[\Omega(s-t)]}\right\}\psi(y)\psi^*(z)\right\}'_z\bigg|_{z=y}
\end{aligned}
$$

$$= X(t, \mathfrak{C}(\psi)) + \frac{1}{2} \frac{\hbar \sin[\Omega(t-s)]}{m\Omega} \int_{-\infty}^{\infty} dz \left\{ -\frac{i}{\hbar} P(s, \mathfrak{C}(\psi))\psi(z)\psi^*(z) \right.$$

$$+ \psi'(z)\psi^*(z) \bigg\} + \frac{1}{2} \frac{\hbar \sin[\Omega(t-s)]}{m\Omega} \int_{-\infty}^{\infty} dy \left\{ \frac{i}{\hbar} P(s, \mathfrak{C}(\psi))\psi(y)\psi^*(y) \right.$$

$$\left. + \psi(y)(\psi^*(y))' \right\}$$

$$= X(t, \mathfrak{C}(\psi)) + \frac{\hbar \sin[\Omega(t-s)]}{m\Omega} \int_{-\infty}^{\infty} dy \{\psi(y)(\psi^*(y))\}' = X(t, \mathfrak{C}(\psi)).$$

Thus, relation (A.2) is valid.

Let us show that

$$\alpha_\Psi^{(0,2)}(t, \hbar) = \alpha^{(0,2)}(t, \hbar, \mathfrak{C}(\psi)). \tag{A.3}$$

Calculate the moment of the second order (A.3). Using the explicit form of the functions $\Psi(x, t)$ we obtain

$$\alpha_\Psi^{(0,2)}(t, \hbar) = \frac{1}{\|\Psi(t)\|^2} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz (\Delta x)^2$$

$$\times \sqrt{\frac{m\Omega}{2\pi i\hbar \sin[\Omega(t-s)]}} \sqrt{\frac{-m\Omega}{2\pi i\hbar \sin[\Omega(t-s)]}}$$

$$\times \exp\left\{ -\frac{i}{\hbar}[S(t, \hbar, \mathfrak{g}(t, \mathfrak{C}(\psi))) - S(s, \hbar, \mathfrak{g}(s, \mathfrak{C}(\psi))) \right.$$

$$+ P(t, \mathfrak{C}(\psi))\Delta x - P(s, \mathfrak{C}(\psi))\Delta z \bigg\}$$

$$\times \exp\left\{ -\frac{im\Omega}{2\hbar} \left( \frac{2\Delta x \Delta z - (\Delta x^2 + \Delta z^2)\cos[\Omega(t-s)]}{\sin[\Omega(s-t)]} \right) \right\}$$

$$\times \exp\left\{ \frac{i}{\hbar}[S(t, \hbar, \mathfrak{g}(t, \mathfrak{C}(\psi))) - S(s, \hbar, \mathfrak{g}(s, \mathfrak{C}(\psi))) \right.$$

$$+ P(t, \mathfrak{C}(\psi))\Delta x - P(s, \mathfrak{C}(\psi))\Delta y \bigg\}$$

$$\times \exp\left\{ \frac{im\Omega}{2\hbar} \left( \frac{2\Delta x \Delta y - (\Delta x^2 + \Delta y^2)\cos[\Omega(s-t)]}{\sin[\Omega(s-t)]} \right) \right\} \psi(y)\psi^*(z)$$

$$= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \frac{m\Omega(\Delta x)^2 \psi(y)\psi^*(z)}{2\pi\hbar \sin[\Omega(t-s)]}$$

$$\times \exp\left\{ -\frac{i}{\hbar}[P(s, \mathfrak{C}(\psi))(\Delta z - \Delta y)] \right\} \exp\left\{ \frac{im\Omega}{\hbar} \left( \frac{\Delta x(\Delta y - \Delta z)}{\sin[\Omega(s-t)]} \right) \right\}$$

$$\times \exp\left\{ \frac{im\Omega}{2\hbar} \left( \frac{(\Delta z^2 - \Delta y^2)\cos[\Omega(s-t)]}{\sin[\Omega(s-t)]} \right) \right\}$$

$$= -\int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} d\omega \frac{\psi(y)\psi^*(z)}{2\pi} \left( \frac{\hbar\omega \sin[\Omega(t-s)]}{m\Omega} \right)^2$$

$$\times \exp\{i\omega(\Delta y - \Delta z)\} \exp\left\{ \frac{i}{\hbar}[P(s, \mathfrak{C}(\psi))(\Delta y - \Delta z)] \right\}$$

$$\times \exp\left\{ \frac{im\Omega}{2\hbar} \frac{(\Delta z^2 - \Delta y^2)\cos[\Omega(s-t)]}{\sin[\Omega(s-t)]} \right\}$$

$$= -\int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dy\, \delta''(y-z)\psi(y)\psi^*(z)\frac{\hbar^2 \sin^2[\Omega(t-s)]}{m^2\Omega^2}$$

$$\times \exp\left\{-\frac{i}{\hbar}[P(s,\mathfrak{C}(\psi))(\Delta z - \Delta y)]\right\}$$

$$\times \exp\left\{\frac{im\Omega}{2\hbar}\left(\frac{(\Delta z^2 - \Delta y^2)\cos[\Omega(s-t)]}{\sin[\Omega(s-t)]}\right)\right\}$$

$$= -\int_{-\infty}^{\infty} dz\left\{\psi(y)\psi^*(z)\frac{\hbar^2 \sin^2[\Omega(t-s)]}{m^2\Omega^2}\exp\left\{-\frac{i}{\hbar}P(s,\mathfrak{C}(\psi))(\Delta z - \Delta y)\right\}\right.$$

$$\left.\times \exp\left\{\frac{im\Omega}{2\hbar}\frac{(\Delta z^2 - \Delta y^2)\cos[\Omega(s-t)]}{\sin[\Omega(s-t)]}\right\}\right\}''_{zz}\bigg|_{y=z}.$$

The expression in braces can be rearranged as follows:

$$\left\{\frac{\hbar^2 \sin^2[\Omega(t-s)]}{m^2\Omega^2}\exp\left\{-\frac{i}{\hbar}[P(s,\mathfrak{C}(\psi))(\Delta z - \Delta y)]\right\}\right.$$

$$\left.\times \exp\left\{\frac{im\Omega}{2\hbar}\left(\frac{(\Delta z^2 - \Delta y^2)\cos[\Omega(s-t)]}{\sin[\Omega(s-t)]}\right)\right\}\psi(y)\psi^*(z)\right\}''_{y=z}$$

$$= -\frac{\hbar^2 \sin^2[\Omega(t-s)]}{m^2\Omega^2}\left\{-\frac{p_0^2}{\hbar^2}\psi(z)\psi^*(z) + \frac{2p_0}{\hbar^2}m\Omega\Delta z\cot[\Omega(t-s)]\psi(z)\psi^*(z)\right.$$

$$-\frac{2i}{\hbar}p_0\psi'(z)\psi^*(z) - \frac{2i}{\hbar}m\Omega\cot[\Omega(s-t)]\Delta z\psi'(z)\psi^*(z) + \psi''(z)\psi^*(z)$$

$$\left.-\frac{1}{\hbar^2}m^2\Omega^2\cot^2\Omega(s-t)]\Delta z^2\psi(z)\psi^*(z) - \frac{im\Omega}{\hbar}\cot[\Omega(s-t)]\psi(z)\psi^*(z)\right\}$$

$$= \frac{p_0^2}{m^2\Omega^2}\sin^2[\Omega(s-t)]\psi(z)\psi^*(z) + \frac{2p_0}{m\Omega}\sin[\Omega(s-t)]$$

$$\times \cos[\Omega(s-t)]\Delta z\psi(z)\psi^*(z) + \frac{2i\hbar p_0}{m^2\Omega^2}\sin^2[\Omega(s-t)]\psi'(z)\psi^*(z)$$

$$+ \frac{2i\hbar}{m\Omega}\sin[\Omega(s-t)]\cos[\Omega(s-t)]\Delta z\psi'(z)\psi^*(z)$$

$$-\frac{\hbar^2}{m^2\Omega^2}\sin^2[\Omega(s-t)]\psi''(z)\psi^*(z) + \cos^2\Omega(s-t)]\Delta z^2\psi(z)\psi^*(z)$$

$$+ \frac{i\hbar}{m\Omega}\sin[\Omega(s-t)]\cos[\Omega(s-t)]\psi(z)\psi^*(z).$$

Then,

$$\alpha_\Psi^{(0,2)}(t,\hbar) = \frac{\sin^2[\Omega(s-t)]}{m^2\Omega^2}\left[p_0^2\int_{-\infty}^{\infty} dz\,\psi^*(z)\psi(z) - 2p_0\int_{-\infty}^{\infty} dz\,\psi^*(z)(-i\hbar)\psi'(z)\right.$$

$$\left.+ \int_{-\infty}^{\infty} dz\,\psi^*(z)(-i\hbar)^2\psi''(z)\right] + \frac{\sin[2\Omega(s-t)]}{m\Omega}\left[p_0\int_{-\infty}^{\infty} dz\,\psi^*(z)\Delta z\psi(z)\right.$$

$$\left.-\int_{-\infty}^{\infty} dz\,\psi^*(z)\Delta z(-i\hbar)\psi'(z) + \frac{i\hbar}{2}\int_{-\infty}^{\infty} dz\psi^*(z)\psi(z)\right]$$

$$+ \cos^2\Omega(s-t)]\int_{-\infty}^{\infty} dz\,\Delta z^2\psi^*(z)\psi(z).$$

In view of the relations

$$\alpha_\psi^{(2,0)} = \int_{-\infty}^{\infty} dz (-i\hbar\partial_z - p_0)^2 \psi^*(z)\psi(z) \qquad \alpha_\psi^{(0,2)} = \int_{-\infty}^{\infty} dz\, \Delta z^2 \psi^*(z)\psi(z)$$

$$\alpha_\psi^{(1,1)} = \int_{-\infty}^{\infty} dz\, \frac{1}{2}[\Delta z(-i\hbar\partial_z - p_0) + (-i\hbar\partial_z - p_0)\Delta z]\psi^*(z)\psi(z)$$

$$= -p_0 \int_{-\infty}^{\infty} dz\, \psi^*(z)\Delta z\psi(z) + \int_{-\infty}^{\infty} dz\, \psi^*(z)\Delta z(-i\hbar)\psi'(z) - \frac{i\hbar}{2}\int_{-\infty}^{\infty} dz\, \psi^*(z)\psi(z)$$

in the case $s = 0$, we obtain

$$\alpha_\Psi^{(0,2)}(t,\hbar) = -\alpha_\psi^{(2,0)}\frac{\sin^2[\Omega(t)]}{m^2\Omega^2} - \alpha_\psi^{(1,1)}\frac{\sin[2\Omega(t)]}{m\Omega} + \alpha_\psi^{(0,2)}\cos^2\Omega(s-t)]$$

$$= \frac{\alpha_\psi^{(1,1)}}{m\Omega}\sin 2\Omega t + \frac{1}{2}\left(\alpha_\psi^{(0,2)} - \frac{\alpha_\psi^{(2,0)}}{m^2\Omega^2}\right)\cos 2\Omega t \frac{1}{2}\left(\alpha_\psi^{(0,2)} + \frac{\alpha_\psi^{(2,0)}}{m^2\Omega^2}\right).$$

From here, in view of formulae (1.19) and (1.27), relation (A.3) follows. The proof of other equalities in (A.1) is similar.

(3.) Let us show that the function $\Psi(x,t)$ satisfies equation (0.1). Substituting (3.5) into (0.1), we obtain

$$\{-i\hbar\partial_t + \widehat{\mathcal{H}}(t) + \varkappa\widehat{V}(t,\Psi)\}\Psi(x,t,\hbar)$$

$$= \int_{-\infty}^{\infty} dy\, \psi(y)\left\{-i\hbar\partial_t + \frac{\hat{p}^2}{2m} + \frac{kx^2}{2} + \frac{\tilde{\varkappa}}{2}\left[ax^2 + 2bx\left[x_\Psi(t,\hbar) + \alpha_\Psi^{(0,1)}(t,\hbar)\right]\right.\right.$$

$$\left.\left. + c\left[x_\Psi^2(t,\hbar) + 2x_\Psi(t,\hbar)\alpha_\Psi^{(0,1)}(t,\hbar) + \alpha_\Psi^{(0,2)}(t,\hbar)\right]\right]\right\}$$

$$\times G_\varkappa(x,y,t,s,\mathfrak{g}(t,\mathfrak{C}(\psi)),\mathfrak{g}_0(\psi))$$

$$= \int_{-\infty}^{\infty} dy\, \psi(y)G_\varkappa(x,y,t,s,\mathfrak{g}(t,\mathfrak{C}(\psi)),\mathfrak{g}_0(\psi))\left\{\frac{i\hbar\Omega}{2}\frac{\cos[\Omega(t-s)]}{\sin[\Omega(t-s)]}\right.$$

$$+ \dot{S}(t,\hbar,\mathfrak{g}(t,\mathfrak{C}(\psi))) + \dot{P}(t,\mathfrak{C}(\psi))\Delta x - P(t,\mathfrak{C}(\psi))\dot{X}(t,\mathfrak{C}(\psi))$$

$$- m\Omega\dot{X}(t,\mathfrak{C}(\psi))\frac{\Delta y - \Delta x\cos[\Omega(t-s)]}{\sin[\Omega(t-s)]}$$

$$- \frac{m\Omega^2}{2}\left(\frac{2\Delta x\Delta y\cos[\Omega(t-s)] - (\Delta x^2 + \Delta y^2)}{\sin^2[\Omega(t-s)]}\right)$$

$$+ \frac{1}{2m}\left[P^2(t,\mathfrak{C}(\psi)) + 2P(t,\mathfrak{C}(\psi))m\Omega\frac{\Delta y - \Delta x\cos[\Omega(t-s)]}{\sin[\Omega(s-t)]}\right.$$

$$\left. + \left(m\Omega\frac{\Delta y - \Delta x\cos[\Omega(t-s)]}{\sin[\Omega(s-t)]}\right)^2\right] - \frac{i\Omega\hbar}{2}\frac{\cos[\Omega(t-s)]}{\sin[\Omega(t-s)]}$$

$$+ \frac{(k + \tilde{\varkappa}a)}{2}(X^2(t,\mathfrak{C}(\psi)) + 2X^2(t,\mathfrak{C}(\psi))\Delta x + \Delta x^2)$$

$$+ \tilde{\varkappa}b(X(t,\mathfrak{C}(\psi)) + \Delta x)\left[x_\Psi(t,\hbar) + \alpha_\Psi^{(0,1)}(t,\hbar)\right]$$

$$+ \frac{\tilde{\varkappa}c}{2}\left[x_\Psi^2(t,\hbar) + 2x_\Psi(t,\hbar)\alpha_\Psi^{(0,1)}(t,\hbar) + \alpha_\Psi^{(0,2)}(t,\hbar)\right]\right\}.$$

Using relations (A.1) (where, in particular, $\alpha_\Psi^{(0,1)}(t, \hbar) = 0$) and the expressions

$$\dot{S}(t, \hbar, \mathfrak{g}(t, \mathfrak{C}(\psi))) = P(t, \mathfrak{C}(\psi))\dot{X}(t, \mathfrak{C}(\psi)) - \frac{P^2(t, \mathfrak{C}(\psi))}{2m} + \frac{kX^2(t, \mathfrak{C}(\psi))}{2}$$

$$+ \tilde{\varkappa}c\sigma_{xx}(t, \hbar, \mathfrak{C}(\psi)) + \frac{\tilde{\varkappa}}{2}(a + 2b + c)X^2(t, \mathfrak{C}(\psi))$$

we obtain

$$\{-\mathrm{i}\hbar\partial_t + \widehat{\mathcal{H}}(t) + \varkappa\widehat{V}(t, \Psi)\}\Psi(x, t, \hbar)$$

$$= \int_{-\infty}^{\infty} \mathrm{d}y\, \psi(y)G_\varkappa(x, y, t, s, \mathfrak{g}(t, \mathfrak{C}(\psi)), \mathfrak{g}_0(\psi))\left\{\dot{P}(t, \mathfrak{C}(\psi))\Delta x\right.$$

$$- \frac{m\Omega^2}{2}(\Delta x)^2 + \frac{(k + \tilde{\varkappa}a)(2X(t, \mathfrak{C}(\psi))\Delta x + (\Delta x)^2)}{2}$$

$$\left. + \tilde{\varkappa}bX^2(t, \mathfrak{C}(\psi))\Delta x\right\} = 0$$

$\square$

## Appendix B

**Proof of theorem 3.2.** Assume the opposite, that is, relation (3.8) is not valid and involves some function $\Phi(x, t, s)$ on the right-hand side. Then, in view of (3.2), relation (3.8) can be represented as $\Phi(x, t, s) = \widehat{U}_\varkappa^{-1}(t, s, \Psi)(x)$, where $\Psi(x, t) = \widehat{U}_\varkappa(t, s, \psi)(x)$. According to the definition of the operator $\widehat{U}_\varkappa^{-1}(t, s, \cdot)$ (3.6), the parameters $\mathfrak{C}(\Psi(t))$ entering in this relation are determined from equation (3.7), which in our case becomes

$$\mathfrak{g}(s, \mathfrak{C})|_{s=t} = \langle\Psi(t)|\hat{\mathfrak{g}}|\Psi(t)\rangle. \tag{B.1}$$

By virtue of theorem 1.1, the relation $\mathfrak{C}(\Psi(t)) = \mathfrak{C}(\psi)$ is valid. Hence, the operator $\widehat{U}_\varkappa^{-1}(t, s, \cdot)$, as it acts on the function $\Psi(x, t)$, and the operator $\widehat{U}_\varkappa^{-1}(t, s, \cdot)$, as it acts on the function $\psi(x)$, are defined on the same trajectory $\mathfrak{g}(t, \mathfrak{C}(\psi))$. Designate $\Delta x = x - X(s, \mathfrak{C}(\psi))$, $\Delta y = y - X(t, \mathfrak{C}(\psi))$, $\Delta z = z - X(s, \mathfrak{C}(\psi))$; then (3.8) can be presented as

$$\Phi(x, t, s) = \widehat{U}_\varkappa^{-1}(t, s, \widehat{U}_\varkappa(t, s, \psi))(x)$$

$$= \int_{-\infty}^{\infty} \mathrm{d}y \int_{-\infty}^{\infty} \mathrm{d}z\, \sqrt{\frac{m\Omega}{2\pi\mathrm{i}\hbar\sin[\Omega(s-t)]}}\sqrt{\frac{m\Omega}{2\pi\mathrm{i}\hbar\sin[\Omega(t-s)]}}$$

$$\times \exp\left\{\frac{\mathrm{i}}{\hbar}[S(s, \hbar, \mathfrak{g}(s, \mathfrak{C}(\psi))) - S(t, \hbar, \mathfrak{g}(\mathfrak{C}(\psi)))\right.$$

$$\left. + P(s, \mathfrak{C}(\psi))\Delta x - P(t, \mathfrak{C}(\psi))\Delta z]\right\}$$

$$\times \exp\left\{\frac{\mathrm{i}m\Omega}{2\hbar}\left(\frac{2\Delta x\Delta z - (\Delta x^2 + \Delta z^2)\cos[\Omega(t-s)]}{\sin[\Omega(t-s)]}\right)\right\}$$

$$\times \exp\left\{\frac{\mathrm{i}}{\hbar}[S(t, \hbar, \mathfrak{g}(t, \mathfrak{C}(\psi))) - S(s, \hbar, \mathfrak{g}(s, \mathfrak{C}(\psi)))\right.$$

$$\left. + P(t, \mathfrak{C}(\psi))\Delta z - P(s, \mathfrak{C}(\psi))\Delta y]\right\}$$

$$\times \exp\left\{\frac{\mathrm{i}m\Omega}{2\hbar}\left(\frac{2\Delta z\Delta y - (\Delta z^2 + \Delta y^2)\cos[\Omega(s-t)]}{\sin[\Omega(s-t)]}\right)\right\}\psi(y)$$

$$= \int_{-\infty}^{\infty} \mathrm{d}y \int_{-\infty}^{\infty} \mathrm{d}z\, \frac{m\Omega}{2\pi\hbar\sin[\Omega(s-t)]}\exp\left\{\frac{\mathrm{i}}{\hbar}[P(s, \mathfrak{C}(\psi))(\Delta x - \Delta y)]\right\}$$

$$\times \exp\left\{\frac{im\Omega}{2\hbar}\left(\frac{2(\Delta x - \Delta y)\Delta z - (\Delta x^2 - \Delta y^2)\cos[\Omega(t-s)]}{\sin[\Omega(t-s)]}\right)\right\}\psi(y)$$

$$= \int_{-\infty}^{\infty} dy\, \delta(x-y) \exp\left\{\frac{i}{\hbar}[P(s, \mathfrak{C}(\psi))(x-y)]\right\}$$

$$\times \exp\left\{\frac{im\Omega}{2\hbar}\left(\frac{(\Delta y^2 - \Delta x^2)\cos[\Omega(t-s)]}{\sin[\Omega(t-s)]}\right)\right\}\psi(y) = \psi(x)$$

and this contradiction proves the theorem. $\square$

## References

[1] Ovsjannikov L V 1982 *Group Analysis of Differential Equations* (New York: Academic)
[2] Meirmanov A M, Pukhnachov V V and Shmarev S I 1994 *Evolution Equations and Lagrangian Coordinates* (Berlin: de Gruyter)
[3] Anderson R L and Ibragimov N H 1979 *Lie–Backlund Transformations in Applications* (Philadelphia, PA: SIAM)
[4] Olver P J 1986 *Application of Lie Groups to Differential Equations* (New York: Springer)
[5] Fushchich W I and Nikitin A G 1987 *Symmetries of Maxwell Equations* (Dordrecht: Reidel)
[6] Gaeta G 1994 *Nonlinear Symmetry and Nonlinear Equations* (Dordrecht: Kluwer)
[7] Khagn Fuk Vo and Chetverikov V M 1978 Generalized solitons of the Schrödinger equation with unitary nonlinearity *Teor. Mat. Fiz.* **36** 345–51
   Khagn Fuk Vo and Chetverikov V M 1978 *Teor. Math. Phys.* **36** (Engl. Transl.)
[8] Belov V V, Trifonov A Yu and Shapovalov A V 2002 The trajectory-coherent approximation and the system of moments for the Hartree type equation *Int. J. Math. Sci.* **32** 325–70
[9] Belov V V, Trifonov A Yu and Shapovalov A V 2002 Semiclassical trajectory-coherent approximation for the Hartree type equation *Teor. Mat. Fiz.* **130** 460–92
   Belov V V, Trifonov A Yu and Shapovalov A V 2002 *Theor. Math. Phys.* **130** 391–418 (Engl. Transl.)
[10] Maslov V P 1977 *The Complex WKB Method for Nonlinear Equations* (Moscow: Nauka)
   Maslov V P 1994 *The Complex WKB Method for Nonlinear Equations: I. Linear Theory* (Basel: Birkhauser) (Engl. Transl.)
[11] Belov V V and Dobrokhotov S Yu 1992 Semiclassical Maslov asymptotics with complex phases: I. General approach *Teor. Mat. Fiz.* **130** 215–54
   Belov V V and Dobrokhotov S Yu 1992 *Theor. Math. Phys.* **92** 843–76 (Engl. Transl.)
[12] Bagrov V G, Belov V V, and Trifonov A Yu 1996 Semiclassical trajectory-coherent approximation in quantum mechanics: I. High order corrections to multidimensional time-dependent equations of Schrödinger type *Ann. Phys., NY* **246** 231–80
[13] Hartree D R 1928 The wave mechanics of an atom with a non-Coulomb central field, part I *Proc. Camb. Phil. Soc.* **24** 89–110
   Hartree D R 1928 The wave mechanics of an atom with a non-Coulomb central field, part II *Proc. Camb. Phil. Soc.* **24** 111–32
   Hartree D R 1928 The wave mechanics of an atom with a non-Coulomb central field, part III *Proc. Camb Phil. Soc.* **24** 426–37
[14] Landau L D and Lifshitz E M 1977 *Quantum Mechanics: Non-Relativistic Theory* (Oxford: Pergamon)
[15] Pitaevskii L P 1988 Bose–Einstein condensation in magnetic traps. Introduction to the theory *Usp. Fiz. Nauk.* **168** 641–53
   Pitaevskii L P 1988 *Phys. Usp.* **41** 569–80 (Engl. Transl.)
[16] Pukhnachov V V 1987 Transformation of equivalence and the latent symmetry of the evolutionary equations *Dokl. AN SSSR* **294** 535–8
   Pukhnachov V V 1987 *Sov. Math. Dokl.* **34** (Engl. Transl.)
[17] Ehrenfest P 1927 Bemerkung über die angenherte Gültigkeit der klassishen Mechanik innerhalb der Quanten Mechanik *Z. Phys.* **45** 455–7
[18] Maslov V P and Fedoriuk M V 1990 *Semiclassical Aproximation in Quantum Mechanics* (Dordrecht: Reidel)
[19] Robertson H P 1934 An indeterminacy relation for several observables and its classical interpretation *Phys. Rev.* **46** 794–801
[20] Dodonov V V and Man'ko V I 1985 Universal invariants of quantum systems and generalized uncertianty relation *Group Theoretical Methods in Physics* vol 1 (London: Harwood Academic) pp 591–612

[21] Belov V V and Kondratyeva M F 1994 Hamiltonian systems of equations for quantum means *Mat. Zametki* **56** 27–39
    Belov V V and Kondratyeva M F 1994 *Math. Notes.* **56** 1228–77 (Engl. Transl)
[22] Malkin M A and Manko V I 1979 *Dynamic Symmetries and Coherent States of Quantum Systems* (Moscow: Nauka) (in Russian)
[23] Perelomov A M 1986 *Generalized Coherent States and Their Application* (Berlin: Springer)
[24] Berry M V 1984 Quantum phase factors accompanying adiabatic changes *Proc. R. Soc.* A **392** 1802 45–58
[25] Anandan J and Aharonov Y 1988 Geometric quantum phase and angles *Phys. Rev.* D **38** 1863–70
[26] Bateman H and Erdelyi A 1953 *Higher Transcendental Functions* vol 2 (London: McGraw-Hill)